The classification of six-dimensional 4-Lie algebras

2009 J. Phys. A: Math. Theor. 42 035207

(http://iopscience.iop.org/1751-8121/42/3/035207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 035207 (17pp) doi:10.1088/1751-8113/42/3/035207

The classification of six-dimensional 4-Lie algebras*

Ruipu Bai and Guojie Song

College of Mathematics and Computer, Hebei University, Baoding 071002, People's Republic of China

E-mail: bairp1@yahoo.com.cn and songguojie821209@yahoo.com.cn

Received 2 July 2008, in final form 10 November 2008
Published 9 December 2008
Online at stacks.iop.org/JPhysA/42/035207

Abstract
This paper concerns properties of $(n+2)$-dimensional n-Lie algebras, and shows the classification of six-dimensional 4-Lie algebras over an algebraically closed field of characteristic zero.

PACS numbers: 02.20.Sv, 02.10.Hh
Mathematics Subject Classification: 17B05, 17D99

1. Introduction

The study of n-Lie algebras has developed a new chapter in the study of Lie theory, attracting many researchers in different areas due largely to the close connections between n-Lie algebras and dynamics, geometries as well as string theory. In the 1970s, describing the simultaneous classical dynamics of three particles as a preliminary step towards a quantum statistic for the quark model, Nambu generalized the Poisson bracket [1] and obtained a three-linear product $\{,,\}$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \{H_1, H_2, x\},$$

where H_1, H_2 are Hamiltonians. In 1994, Takhtajan [2] developed the geometrical ideas of Nambu mechanics, and introduced the fundamental identity, an analogue of the Jacobi identity. This allowed him to establish the connections between the generalized Nambu mechanics and the theory of n-Lie algebras proposed by Filippov [3].

In recent years, much work has been done in applications of n-Lie algebras. Bagger and Lambert [4] proposed a field theory model for multiple M2-branes based on metric n-Lie algebras. They obtained properties of symmetries that arise from the triple product of an algebra, and then established a supersymmetric theory that is consistent with almost all symmetries. Focusing on the fundamental identity and the relationship with the

* Project partially supported by NSF(10871192) of China, NSF(A2007000138) of Hebei Province and NSF(y2004034) of Hebei University, China.

Nambu–Poisson bracket, Ho*et al* [5] found new 3-Lie algebras and applications in membrane, including the Basu–Harvey equation and the Bagger–Lambert model. More applications can be found in [6–13].

Filippov [3] introduced the concept of n-Lie algebra and classified $(n + 1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. The structure of n-Lie algebras is very different from that of Lie algebras due to the n-ary multilinear operation. For example, up to isomorphism there is a unique finite-dimensional simple n-Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero [14], which is the $(n + 1)$-dimensional n-Lie algebra with a multiplication table

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{i+1} e_i, \qquad 1 \leqslant i \leqslant n + 1$$

in a basis $e_1, \ldots, e_{n+1}$, where symbol $\hat{e}_i$ means that e_i is omitted in the bracket. So far, the only known infinite-dimensional simple n-Lie algebras over fields of characteristic $p \geqslant 0$ are Jacobian algebras and their quotient algebras [15, 16]. Bai and her collaborators [17] showed that there exist only $\left[\frac{n}{2}\right]+1$ classes of $(n+1)$-dimensional simple n-Lie algebras over a complete field of characteristic 2. They also showed that there are no simple $(n + 2)$-dimensional n-Lie algebras.

The purpose of this paper is to classify six-dimensional 4-Lie algebras over an algebraically closed field of characteristic zero. The organization for the remainder of this paper is as follows. Section 2 introduces some basic notions. Section 3 is devoted to properties of $(n + 2)$-dimensional n-Lie algebras and to the classification of six-dimensional 4-Lie algebras.

2. Fundamental notions

An n-Lie algebra is a vector space A over a field F ($\mathrm{char}(F) \neq 2$) equipped with an n-multilinear operation $[x_1, \ldots, x_n]$ satisfying

$$[x_1, \ldots, x_n] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \tag{2.1}$$

and

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] \tag{2.2}$$

for any $x_1, \ldots, x_n, y_2, \ldots, y_n \in A$ and any permutation $\sigma \in S_n$. If $\mathrm{char}(F) = 2$, then identity (2.1) is replaced by

$$[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] = 0$$

for any $x_1, \ldots, x_n \in A$ where $x_i = x_j$ for some $1 \leqslant i < j \leqslant n$. Identity (2.2) is usually called the generalized Jacobi identity, or simply the Jacobi identity.

A derivation of an n-Lie algebra A is a linear map D of A into itself satisfying

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, D(x_i), \ldots, x_n] \tag{2.3}$$

for any $x_1, \ldots, x_n \in A$. Let $\mathrm{Der}(A)$ be the set of all derivations of A. Then $\mathrm{Der}(A)$ is a Lie subalgebra of the general linear Lie algebra $gl(A)$ and is called the derivation algebra of A. The map $\mathrm{ad}(x_1, \ldots, x_{n-1})$: $A \to A$, given by

$$\mathrm{ad}(x_1, \ldots, x_{n-1})(x_n) = [x_1, \ldots, x_n], \qquad \text{for} \quad x_n \in A,$$

is referred to as a left multiplication defined by elements $x_1, \ldots, x_{n-1} \in A$. It follows from identity (2.2), that $\mathrm{ad}(x_1, \ldots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of Der(A), which we denote by ad(A). Every derivation in ad(A) is by definition an inner derivation.

If a subspace B of an n-Lie algebra A satisfying $[x_1, \ldots, x_n] \in B$ for any $x_1, \ldots, x_n \in B$, then B is called a subalgebra of A. Let $A_1, A_2, \ldots, A_n$ be subalgebras of an n-Lie algebra A. Denote by $[A_1, A_2, \ldots, A_n]$ the subspace of A generated by all vectors $[x_1, \ldots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra $A^1 = [A, A, \ldots, A]$ is called the derived algebra of A. If $A^1 = 0$, then A is called an Abelian n-Lie algebra.

Let H be an Abelian subalgebra of n-Lie algebra A. Then H is by definition a Toral subalgebra of A, if A is a complete H-module, that is

$$A = \oplus_{\alpha \in (H^{n-1})^*} A_\alpha \text{ (direct sum as vector spaces)},$$

where

$$A_\alpha = \{x \in A | \mathrm{ad}(h_1, \ldots, h_{n-1})(x) = \alpha(h_1, \ldots, h_{n-1})(x), \forall\, (h_1, h_2, \ldots, h_{n-1}) \in H^{n-1}\}.$$

A Toral subalgebra H is called maximal if there are no Toral subalgebras of A properly containing H. An ideal I of an n-Lie algebra A is a subspace of A such that $[I, A, \ldots, A] \subseteq I$. If $[I, I, A, \ldots, A] = 0$, then I is referred to as an Abelian ideal. If $A^1 \neq 0$ and A has no ideals except 0 and itself, then A is by definition a simple n-Lie algebra. An n-Lie algebra A is said to be decomposable if there are nonzero ideals I_1, I_2 such that

$$A = I_1 \oplus I_2,$$

then $[I_1, I_2, A, \ldots, A] = 0$. Otherwise, we say that A is indecomposable. Clearly if A is a simple n-Lie algebra then A is indecomposable.

The subset $Z(A) = \{x \in A | [x, y_1, \ldots, y_{n-1}] = 0, \forall y_1, \ldots, y_{n-1} \in A\}$ is called the center of A. It is clear that $Z(A)$ is an Abelian ideal of A. An ideal I of an n-Lie algebra A is called an n-solvable ideal if I satisfying $I^{(r,n)} = 0$ for some $r \geqslant 0$, where $I^{(0,n)} = I$ and $I^{(r,n)}$ is defined by induction, $I^{(s+1,n)} = [I^{(s,n)}, \ldots, I^{(s,n)}]$ for $s \geqslant 0$. The maximal n-solvable ideal of A is called the n-radical of A. If the only n-solvable ideal of A is zero, then A is by definition a strong semisimple n-Lie algebra.

Lemma 2.1 [14]. *Let A be a finite-dimensional n-Lie algebra over an algebraically closed field of characteristic zero. Then A has a decomposition $A = S \oplus K$, where S is a strong semisimple subalgebra and K is the n-radical of A.*

Lemma 2.2 [18]. *Let A be a finite-dimensional n-Lie algebra over an algebraically closed field of characteristic zero. Then A is strong semisimple if and only if A can be decomposed into the direct sum of its simple ideals.*

3. Classification of six-dimensional 4-Lie algebras

In this section, unless stated otherwise, we suppose that F is an algebraically closed field of characteristic 0. Any brackets of basis vectors not listed in the multiplication table of n-Lie algebras are assumed to be zero.

First, we give the classification of $(n+1)$-dimensional n-Lie algebras over F. It is a simple refinement of the classification given by Fillipov [3].

Theorem 3.1. *Let A be an $(n+1)$-dimensional n-Lie algebra over F and $e_1, e_2, \ldots, e_{n+1}$ be a basis of A $(n \geqslant 3)$. Then one and only one of the following possibilities holds up to isomorphism:*

(a) If $\dim A^1 = 0$*, then A is an Abelian n-Lie algebra.*

(b) If $\dim A^1 = 1$ *and let* $A^1 = Fe_1$*, then in case* $A^1 \subseteq Z(A)$*,*

$$(b_1)[e_2, \ldots, e_{n+1}] = e_1, \tag{3.1}$$

in case A^1 *is not contained in* $Z(A)$*,*

$$(b_2)[e_1, \ldots, e_n] = e_1. \tag{3.2}$$

(c) If $\dim A^1 = 2$ *and let* $A^1 = Fe_1 + Fe_2$*, then*

$$(c_1) \quad \begin{cases}[e_1, e_3, \ldots, e_{n+1}] = e_2,\\ [e_2, \ldots, e_{n+1}] = e_1,\end{cases} \qquad (c_2) \quad \begin{cases}[e_1, e_3, \ldots, e_{n+1}] = e_2,\\ [e_2, \ldots, e_{n+1}] = e_1 + \beta e_2,\end{cases} \tag{3.3}$$

where $\beta \in F$ *and* $\beta \neq 0$.

(d) If $\dim A^1 = r$, $3 \leqslant r \leqslant n+1$*, let* $A^1 = Fe_1 + Fe_2 + \cdots + Fe_r$*. Then*

$$(d_r)[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = e_i, 1 \leqslant i \leqslant r, \tag{3.4}$$

where symbol $\hat{e}_i$ *means that* e_i *is omitted.*

Proof. It suffices to prove cases (c_1) and (d) since the other cases are proved in [3].

If A is the case $(c_1)'\begin{cases}[e_2,\ldots,e_{n+1}]=\alpha e_1,\\ [e_1,e_3,\ldots,e_{n+1}]=e_2,\end{cases}$ $\alpha \in F, \alpha \neq 0$, then substituting $\frac{e_{n+1}}{\sqrt{\alpha}}$ and $\frac{e_2}{\sqrt{\alpha}}$ for e_{n+1} and e_2, respectively, we get (c_1).

If $\dim A^1 = r$, $3 \leqslant r \leqslant n+1$, then from [3] only one of the following cases holds up to isomorphism:

$$(d_r)'[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = \beta_i e_i, \qquad \beta_i \in F, \qquad \beta_i \neq 0, \qquad 1 \leqslant i \leqslant r.$$

If $r \leqslant n$, substituting $\sqrt{\beta_i} e_i$ for e_i, $1 \leqslant i \leqslant r$, and substituting $\frac{e_{n+1}}{\sqrt{\beta_1 \cdots \beta_r}}$ for e_{n+1}, we get (d_r).

If $r = n+1$, then table $(d_{n+1})'$ can be reduced to

$$(d_{n+1})''[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = \alpha e_i, \qquad 1 \leqslant i \leqslant n+1$$

by substituting $\sqrt{\beta_i} e_i$ for e_i, $i = 1, \ldots, n+1$, where $\alpha = \sqrt{\beta_1 \ldots \beta_{n+1}} \neq 0$. Again replacing e_i by $\left(\frac{1}{\alpha}\right)^{\frac{1}{n-1}} e_i$, we get (d_{n+1}). □

Lemma 3.1 [19]. *Let A be an* $(n+2)$*-dimensional n-Lie algebra over an algebraically closed field. Then there exists a subalgebra of A with codimension 1.*

Lemma 3.2. *Let A be an* $(n+2)$*-dimensional n-Lie algebra over F. Then we have* $\dim A^1 \leqslant n+1$.

Proof. From lemma 2.1, A has a decomposition as follows:

$$A = S \oplus K,$$

where S is zero or a strong semisimple n-Lie algebra, K is the n-radical of A. If $S = 0$, then $A = K$ is an n-solvable n-Lie algebra. Therefore, $A^1 = [A, \ldots, A] = [K, \ldots, K]$ is a proper subalgebra of A. The result holds.

If $S \neq 0$, then the dimension of S is equal to $m(n+1)$ by lemma 2.2, where m is a positive integer. Since $\dim A = n+2$, $\dim S = n+1$ and $\dim K = 1$. Hence K is a one-dimensional S-module and $[K, S, \ldots, S] = 0$. Therefore

$$A^1 = [A, \ldots, A] = [S, A, \ldots, A] + [K, A, \ldots, A] = [S, \ldots, S] = S,$$

so A^1 is a proper subalgebra of A. □

Lemma 3.3. *Let A be a non-Abelian $(n+2)$-dimensional n-Lie algebra over F. If* $\dim A^1 \neq 3$, *then there exists a non-Abelian subalgebra of codimension 1 containing A^1.*

Proof. If $\dim A^1 \leqslant 2$, then the result holds by [17]. When $\dim A^1 = r > 3$, from lemma 3.2 we have $r \leqslant n+1$. We suppose that A does not contain a non-Abelian subalgebra with codimension 1 containing A^1. Set $A^1 = Fe_1 + \cdots + Fe_r$. Then the multiplication table of A in the basis $e_1, \ldots, e_r, \ldots, e_{n+2}$ is as follows:

$$\left[e_{i_1}, e_{i_2}, \ldots, e_{i_{r-2}}, e_{r+1}, \ldots, e_{n+2}\right] = b^1_{i_1,\ldots,i_{r-2}} e_1 + \cdots + b^r_{i_1,\ldots,i_{r-2}} e_r, \qquad 1 \leqslant i_1, \ldots, i_{r-2} \leqslant r,$$

and the rank of the matrix

$$B = \begin{pmatrix} b^1_{1,2,\ldots,(r-2)} & b^2_{1,2,\ldots,(r-2)} & \cdots & b^r_{1,2,\ldots,(r-2)} \\ b^1_{1,3,\ldots,(r-1)} & b^2_{1,3,\ldots,(r-1)} & \cdots & b^r_{1,3,\ldots,(r-1)} \\ \vdots & \vdots & \vdots & \vdots \\ b^1_{3,\ldots,r} & b^2_{3,\ldots,r} & \cdots & b^r_{3,\ldots,r} \end{pmatrix}$$

is equal to r, where $b^j_{i_1,\ldots,i_{r-2}} \in F$ for $1 \leqslant i_1, \ldots, i_{r-2}, j \leqslant r$.

Therefore, we get an $(r-2)$-Lie algebra A_0 with $(n-2)$-ary operation $[, \ldots,]_0$ as follows:

$$[x_1, \ldots, x_{r-2}]_0 = [x_1, \ldots, x_{r-2}, e_{r+1}, \ldots, e_{n+2}], \qquad x_1, \ldots, x_{r-2} \in A_0, \tag{3.5}$$

where $A_0 = A$ as vector spaces. It is trivial that the center $Z(A_0) = Fe_{r+1} + \cdots + Fe_{n+2}$. By lemma 2.1 and (3.5), A_0 has a decomposition $A_0 = S \oplus Z(A_0)$, where $S = A_0^1$ is a strong semisimple $(r-2)$-Lie algebra and $\dim S = r$. This contradicts lemma 3.2. Therefore, the result is true. □

Theorem 3.2. *Let A be a six-dimensional 4-Lie algebra over F with a basis $e_1, \ldots, e_6$. Then one and only one of the following possibilities holds up to isomorphism:*

(a) If $\dim A^1 = 0$, *A is Abelian.*
(b) If $\dim A^1 = 1$ *and* $A^1 = Fe_1$, *then*
$(b^1)[e_2, e_3, e_4, e_5] = e_1$, $(b^2)[e_1, e_2, e_3, e_4] = e_1$.
(c) If $\dim A^1 = 2$ *and* $A^1 = Fe_1 + Fe_2$, *then*

$$(c^1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 \\ [e_1, e_3, e_4, e_5] = e_2, \end{cases} \qquad (c^2) \quad \begin{cases} [e_1, e_3, e_4, e_5] = e_2, \\ [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_4, e_5, e_6] = e_1, \\ [e_2, e_4, e_5, e_6] = e_2, \end{cases}$$

$$(c^3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \end{cases} \qquad (c^4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_3, e_4, e_5, e_6] = e_1, \end{cases}$$

$$(c^5) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_2, e_4, e_5, e_6] = e_2, \\ [e_1, e_4, e_5, e_6] = e_1, \end{cases} \qquad (c^6) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_2, \end{cases}$$

$$(c^7) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_4, e_5, e_6] = e_1, \\ [e_2, e_4, e_5, e_6] = e_2, \end{cases} \qquad \textit{where} \quad \beta \in F, \quad \textit{and} \quad \beta \neq 0.$$

(d) If $\dim A^1 = 3$*, let* $A^1 = Fe_1 + Fe_2 + Fe_3$*. Then*

$$(d^1)\quad \begin{cases}[e_1, e_4, e_5, e_6] = e_1 + \alpha e_2,\\ [e_2, e_4, e_5, e_6] = e_2,\\ [e_3, e_4, e_5, e_6] = \beta e_3,\end{cases} \qquad (d^2)\quad \begin{cases}[e_1, e_4, e_5, e_6] = e_1,\\ [e_2, e_4, e_5, e_6] = \alpha e_2,\\ [e_3, e_4, e_5, e_6] = \beta e_3,\end{cases}$$

$$(d^3)\quad \begin{cases}[e_1, e_4, e_5, e_6] = e_1 + \alpha e_2,\\ [e_2, e_4, e_5, e_6] = e_2 + \alpha e_3,\\ [e_3, e_4, e_5, e_6] = e_3,\end{cases} \qquad (d^4)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_2, e_4, e_5, e_6] = -e_2,\\ [e_3, e_4, e_5, e_6] = e_3,\end{cases}$$

$$(d^5)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_3, e_4, e_5, e_6] = e_3 + \alpha e_2,\\ [e_2, e_4, e_5, e_6] = e_3,\\ [e_1, e_4, e_5, e_6] = e_1,\end{cases} \qquad (d^6)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_3, e_4, e_5, e_6] = e_3,\\ [e_2, e_4, e_5, e_6] = \gamma e_2,\\ [e_1, e_4, e_5, e_6] = (\gamma + 1)e_1\end{cases}$$

$$(d^7)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, e_3, e_4, e_5] = e_2,\\ [e_1, e_2, e_4, e_5] = e_3,\end{cases} \qquad \textit{where}\quad \alpha, \beta, \gamma \in F, \alpha\beta\gamma \neq 0 \quad \textit{and} \quad \gamma \neq -1.$$

(e) If $\dim A^1 = 4$*, let* $A^1 = Fe_1 + Fe_2 + Fe_3 + Fe_4$*. Then*

$$(e^1)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, e_3, e_4, e_5] = e_2,\\ [e_1, e_2, e_4, e_5] = e_3,\\ [e_1, e_2, e_3, e_5] = e_4,\end{cases} \qquad (e^2)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_3, e_4, e_5, e_6] = e_2,\\ [e_2, e_4, e_5, e_6] = e_3,\\ [e_2, e_3, e_5, e_6] = e_4.\end{cases}$$

(f) If $\dim A^1 = 5$*, let* $A^1 = Fe_1 + Fe_2 + Fe_3 + Fe_4 + Fe_5$*. Then*

$$(f^1)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, e_3, e_4, e_5] = e_2,\\ [e_1, e_2, e_4, e_5] = e_3,\\ [e_1, e_2, e_3, e_5] = e_4,\\ [e_1, e_2, e_3, e_4] = e_5,\end{cases} \qquad (f^2)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_3, e_4, e_5, e_6] = e_2,\\ [e_2, e_4, e_5, e_6] = e_3,\\ [e_2, e_3, e_5, e_6] = e_4,\\ [e_2, e_3, e_4, e_6] = e_5.\end{cases}$$

Proof. (1) Case (a) is trivial.

(2) Case (b). Suppose $A^1 = Fe_1$. Then from lemma 3.3 and theorem 3.1, the multiplication table of A in the basis $e_1, \ldots, e_6$ has the following possibilities:

$$(1)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = b_{ij}e_1, \qquad b_{ij} \in F, \qquad 1 \leqslant i < j \leqslant 5,\end{cases}$$

$$(2)\quad \begin{cases}[e_1, e_2, e_3, e_4] = e_1,\\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = b_{ij}e_1, \qquad b_{ij} \in F, \qquad 1 \leqslant i < j \leqslant 5.\end{cases}$$

Substituting $e_1 = [e_2, \ldots, e_{n+1}]$ into other equations of (1) and applying Jacobi identities for $\{e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6\}$, $1 < i < j \leqslant 5$ we get

$$b_{ij}e_1 = [[e_2, e_3, e_4, e_5], \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = 0.$$

It follows that $b_{ij} = 0$ for $1 < i < j \leqslant 5$. Then table (1) is in the form of

$$(1)' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_2, e_4, e_5, e_6] = b_{13}e_1, \\ [e_2, e_3, e_5, e_6] = b_{14}e_1, \\ [e_2, e_3, e_4, e_6] = b_{15}e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}e_1. \end{cases}$$

If substituting $e_6 - b_{15}e_5 + b_{14}e_4 - b_{13}e_3 + b_{12}e_2$ for e_6 in $(1)'$, then we get

$$(b^1) \quad [e_2, e_3, e_4, e_5] = e_1.$$

Table (2) can be reduced to

$$(2)' \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_1, e_2, e_5, e_6] = b_{34}e_1, \\ [e_1, e_3, e_5, e_6] = b_{24}e_1, \\ [e_1, e_4, e_5, e_6] = b_{23}e_1, \\ [e_2, e_3, e_5, e_6] = b_{14}e_1, \\ [e_2, e_4, e_5, e_6] = b_{13}e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}e_1, \end{cases}$$

by substituting $e_6 + b_{15}e_1 - b_{25}e_2 + b_{35}e_3 - b_{45}e_4$ for e_6 in table (2).

Applying the Jacobi identities for quadruple $\{b_{1j}e_1,\ e_2,\ e_3,\ e_4\}$, $j = 2, 3, 4$, we obtain $b_{12} = b_{13} = b_{14} = 0$. Furthermore, since

$$0 = [[e_2, e_3, e_4, e_6], e_1, e_4, e_5] = b_{23}e_1, \qquad 0 = [[e_2, e_3, e_4, e_6], e_1, e_3, e_5] = b_{24}e_1,$$
$$0 = [[e_2, e_3, e_4, e_6], e_1, e_2, e_5] = b_{34}e_1,$$

we get that table (2) is isomorphic to (b^2). And by theorem 3.1, (b^1) is not isomorphic to (b^2).

(3) If $\dim A^1 = 2$, let $A^1 = Fe_1 + Fe_2$. Then the multiplication table of A in the basis $e_1, \ldots, e_6$ has the following possibilities:

$$(1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{2} b_{ij}^k e_k, \end{cases}$$

$$(2) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{2} b_{ij}^k e_k, \end{cases}$$

$$(3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{2} b_{ij}^k e_k, \end{cases}$$

$$(4) \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{2} b_{ij}^k e_k, \end{cases}$$

where $\beta, b_{ij}^k \in F, \beta \neq 0, 1 \leqslant i < j \leqslant 5$.

First, we impose Jacobi identities on table (1) for quadruple $\{e_2, e_4, e_5, e_6\}$ and $\{e_1, e_2, e_5, e_6\}$ and substitute $e_6 + \sum_{j=2}^{5}(-1)^j b_{1j}^1 e_j + b_{12}^2 e_1$ for e_6. Table (1) is reduced to

$$(1)' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_2, e_4, e_5, e_6] = b_{13}^2 e_2, \\ [e_2, e_3, e_5, e_6] = b_{14}^2 e_2, \\ [e_2, e_3, e_4, e_6] = b_{15}^2 e_2, \\ [e_1, e_4, e_5, e_6] = b_{13}^2 e_1, \\ [e_1, e_3, e_5, e_6] = b_{14}^2 e_1, \\ [e_1, e_3, e_4, e_6] = b_{15}^2 e_1. \end{cases}$$

If $b_{13}^2 = b_{14}^2 = b_{15}^2 = 0$, then (1) is isomorphic to $(c^1)\begin{cases}[e_2,e_3,e_4,e_5]=e_1, \\ [e_1,e_3,e_4,e_5]=e_2.\end{cases}$

If there exists $j\,(3 \leqslant j \leqslant 5)$ such that $b_{1j}^2 \neq 0$, then we may as well suppose $b_{13}^2 \neq 0$. After substituting $e_3 + \sum_{j=4}^{5}(-1)^{j+1}\frac{b_{1j}^2}{b_{13}^2}e_j$ and $\frac{1}{b_{13}^2}e_6$ for e_3 and e_6, table (1) is isomorphic to

$$(c^2) \begin{cases} [e_1, e_3, e_4, e_5] = e_2, \\ [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_4, e_5, e_6] = e_1, \\ [e_2, e_4, e_5, e_6] = e_2. \end{cases}$$

Second, if we apply the Jacobi identities to table (2) and substitute $e_6 + b_{12}^2 e_1 + \sum_{j=3}^{5}(-1)^j b_{2j}^2 e_j$ for e_6, then (2) is reduced to

$$(2)' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_3, e_4, e_5, e_6] = b_{12}^1 e_1, \\ [e_2, e_4, e_5, e_6] = b_{23}^1 e_2, \\ [e_2, e_3, e_5, e_6] = b_{24}^1 e_2, \\ [e_2, e_3, e_4, e_6] = b_{25}^1 e_2, \\ [e_1, e_4, e_5, e_6] = b_{23}^1 e_1, \\ [e_1, e_3, e_5, e_6] = b_{24}^1 e_1, \\ [e_1, e_3, e_4, e_6] = b_{25}^1 e_1, \end{cases} \qquad \beta \in F, \beta \neq 0.$$

If $b_{12}^1 = b_{23}^1 = b_{24}^1 = b_{25}^1 = 0$, then $(2)'$ is in the form of

$$(c^3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \end{cases} \qquad \beta \in F, \beta \neq 0.$$

If $b_{12}^1 = 0$ (the discussion for $b_{12}^1 \neq 0$ is similar) and there exists $k \geqslant 3$ such that $b_{2k}^1 \neq 0$, say $b_{23}^1 \neq 0$, then by substituting $e_3 + \sum_{j=4}^{5}(-1)^{j+1}\frac{b_{2j}^1}{b_{23}^1}e_j - \frac{b_{12}^1}{b_{23}^1}e_1$ and $\frac{1}{b_{23}^1}e_6$ for e_3 and e_6 in $(2)'$, table (2) is isomorphic to

$$(c^5) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_2, e_4, e_5, e_6] = e_2, \\ [e_1, e_4, e_5, e_6] = e_1, \end{cases} \qquad \beta \in F, \beta \neq 0.$$

When $b_{12}^1 \neq 0$, and $b_{23}^1 = b_{24}^1 = b_{25}^1 = 0$, then replacing e_6 by $\frac{1}{b_{12}^1} e_6$ in $(2)'$, we get

$$(c^4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \qquad \beta \in F, \beta \neq 0. \\ [e_3, e_4, e_5, e_6] = e_1, \end{cases}$$

Third, by a similar discussion to that for table (2) and using suitable linear transformations for basis vectors, we get that table (3) is isomorphic to (c^6) or (c^7).

Lastly we discuss table (4). If we replace e_6 by $e_6 + \sum_{j=1}^{4} (-1)^{j+1} b_{j5}^1 e_j$, then (4) is reduced to

$$(4)' \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^1 e_1 + b_{12}^2 e_2, \\ [e_2, e_4, e_5, e_6] = b_{13}^1 e_1 + b_{13}^2 e_2, \\ [e_2, e_3, e_5, e_6] = b_{14}^1 e_1 + b_{14}^2 e_2, \\ [e_2, e_3, e_4, e_6] = b_{15}^2 e_2, \\ [e_1, e_4, e_5, e_6] = b_{23}^1 e_1 + b_{23}^2 e_2, \\ [e_1, e_3, e_5, e_6] = b_{24}^1 e_1 + b_{24}^2 e_2, \\ [e_1, e_3, e_4, e_6] = b_{25}^2 e_2, \\ [e_1, e_2, e_5, e_6] = b_{34}^1 e_1 + b_{34}^2 e_2, \\ [e_1, e_2, e_4, e_6] = b_{35}^2 e_2, \\ [e_1, e_2, e_3, e_6] = b_{45}^2 e_2. \end{cases}$$

By substituting $e_1 = [e_1, e_2, e_3, e_4]$ into the other equations of $(4)'$ and imposing the Jacobi identities, we get

$$(4)'' \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^1 e_1, \\ [e_2, e_4, e_5, e_6] = b_{13}^1 e_1, \\ [e_2, e_3, e_5, e_6] = b_{14}^1 e_1, \\ [e_1, e_4, e_5, e_6] = b_{23}^1 e_1, \\ [e_1, e_3, e_5, e_6] = b_{24}^1 e_1, \\ [e_1, e_2, e_5, e_6] = b_{34}^1 e_1. \end{cases}$$

It follows that $\dim A^1 = 1$. This contradicts $\dim A^1 = 2$. Therefore table (4) is not realized.

Now we need to prove that cases (c^i), $i = 1, \ldots, 7$, represent distinct classes. It is trivial that case (c^i) is not isomorphic to case (c^j) for $i = 1, 3, 6$ and $j = 2, 4, 5, 7$. From theorem 3.1 the table (c^1) is not isomorphic to (c^3). And also (c^1) and (c^3) are not isomorphic to (c^6) since (c^6) is indecomposable. Comparing the dimensions of maximal Toral subalgebras and the dimensions of derivation algebras of cases (c^2), (c^4), (c^5) and (c^7), we get that (c^j) is not isomorphic to $(c^{j'})$ when $j \neq j'$.

(4) If $\dim A^1 = 3$, let $A^1 = Fe_1 + Fe_2 + Fe_3$. Then when there does not exist a five-dimensional non-Abelian subalgebra of A containing A^1, the multiplication table of A in a basis $e_1, \ldots, e_6$ has the following possibilities:

$$\begin{cases} [e_3, e_4, e_5, e_6] = b_{12}^1 e_1 + b_{12}^2 e_2 + b_{12}^3 e_3, \\ [e_2, e_4, e_5, e_6] = b_{13}^1 e_1 + b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_1, e_4, e_5, e_6] = b_{23}^1 e_1 + b_{23}^2 e_2 + b_{23}^3 e_3, \end{cases}$$

where the rank of the matrix

$$M = \begin{pmatrix} b_{12}^1 & b_{12}^2 & b_{12}^3 \\ b_{13}^1 & b_{13}^2 & b_{13}^3 \\ b_{23}^1 & b_{23}^2 & b_{23}^3 \end{pmatrix}$$

is equal to 3.

It is easily proved that two such 4-Lie algebras determined by matrices M and M_1 are isomorphic if and only if there exists a nonsingular (3×3) matrix T and a nonzero element $\alpha \in F$ such that $M = \alpha T M_1 T^{-1}$. Therefore, we get that one and only one of the following possibilities holds up to isomorphism:

$$(d^1) \quad \begin{cases} [e_1, e_4, e_5, e_6] = e_1 + \alpha e_2, \\ [e_2, e_4, e_5, e_6] = e_2, \\ [e_3, e_4, e_5, e_6] = \beta e_3, \end{cases} \qquad (d^2) \quad \begin{cases} [e_1, e_4, e_5, e_6] = e_1, \\ [e_2, e_4, e_5, e_6] = \alpha e_2, \\ [e_3, e_4, e_5, e_6] = \beta e_3, \end{cases}$$

$$(d^3) \quad \begin{cases} [e_1, e_4, e_5, e_6] = e_1 + \alpha e_2, \\ [e_2, e_4, e_5, e_6] = e_2 + \alpha e_3, \\ [e_3, e_4, e_5, e_6] = e_3, \end{cases} \qquad \text{where} \qquad \alpha, \beta \in F \text{ and } \alpha\beta \neq 0.$$

When there exists a five-dimensional non-Abelian subalgebra B containing A^1, we may as well suppose $B = Fe_1 + Fe_2 + Fe_3 + Fe_4 + Fe_5$. Then the multiplication table in the basis $e_1, \ldots, e_6$ of A has only the following possibilities:

$$(1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^3 b_{ij}^k e_k, \end{cases}$$

$$(2) \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^3 b_{ij}^k e_k, \end{cases}$$

$$(3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^3 b_{ij}^k e_k, \end{cases}$$

$$(4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^3 b_{ij}^k e_k, \end{cases}$$

$$(5) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^3 b_{ij}^k e_k, \end{cases}$$

where $b_{ij}^k \in F$, $1 \leqslant i < j \leqslant 5$.

First, by imposing Jacobi identities on table (1) for $\{e_1, e_4, e_5, e_6\}$, $\{e_1, e_3, e_5, e_6\}$, $\{e_1, e_3, e_4, e_6\}$, $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_2, e_4, e_6\}$, $\{e_1, e_2, e_3, e_6\}$, and substituting $e_6 + b^1_{12}e_2 - b^1_{13}e_3 + b^1_{14}e_4 - b^1_{15}e_5$ for e_6, we get

$$(1)' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = b^2_{12}e_2 + b^3_{12}e_3, \\ [e_2, e_4, e_5, e_6] = b^2_{13}e_2 + b^3_{13}e_3, \\ [e_2, e_3, e_5, e_6] = b^2_{14}e_2 + b^3_{14}e_3, \\ [e_2, e_3, e_4, e_6] = b^2_{15}e_2 + b^3_{15}e_3, \\ [e_1, e_4, e_5, e_6] = \left(b^2_{13} + b^3_{12}\right)e_1, \\ [e_1, e_3, e_5, e_6] = b^2_{14}e_1, \\ [e_1, e_3, e_4, e_6] = b^2_{15}e_1, \\ [e_1, e_2, e_5, e_6] = -b^3_{14}e_1, \\ [e_1, e_2, e_4, e_6] = -b^3_{15}e_1, \end{cases}$$

where

$$\det\begin{pmatrix} b^2_{12} & b^3_{12} \\ b^2_{14} & b^3_{14} \end{pmatrix} = 0, \qquad \det\begin{pmatrix} b^2_{12} & b^3_{12} \\ b^2_{15} & b^3_{15} \end{pmatrix} = 0, \qquad \det\begin{pmatrix} b^2_{14} & b^3_{14} \\ b^2_{15} & b^3_{15} \end{pmatrix} = 0,$$

$$\det\begin{pmatrix} b^2_{13} & b^3_{13} \\ b^2_{14} & b^3_{14} \end{pmatrix} = 0 \qquad \text{and} \qquad \det\begin{pmatrix} b^2_{13} & b^3_{13} \\ b^2_{15} & b^3_{15} \end{pmatrix} = 0.$$

Since $\dim A^1 = 3$, we have

$$b^2_{14} = b^3_{14} = b^2_{15} = b^3_{15} = 0, \qquad \det\begin{pmatrix} b^2_{12} & b^3_{12} \\ b^2_{13} & b^3_{13} \end{pmatrix} \neq 0.$$

If $b^3_{12} = b^2_{13} = 0$, replacing e_6 by $\frac{1}{b^2_{12}}e_6$ in $(1)'$, we get

$$(1)'' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_2, \\ [e_2, e_4, e_5, e_6] = \alpha e_3, \end{cases} \qquad \alpha \in F, \alpha \neq 0.$$

Again replacing e_2, e_3, e_6 and e_1 by $e_2-\sqrt{\alpha}e_3$, $e_2+\sqrt{\alpha}e_3$, $\frac{1}{\sqrt{\alpha}}e_6$ and $2\sqrt{\alpha}e_1$ in $(1)''$ respectively, we get that table (1) is isomorphic to

$$(d^4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_2, e_4, e_5, e_6] = -e_2, \\ [e_3, e_4, e_5, e_6] = e_3. \end{cases}$$

If $b^3_{12} = 0, b^2_{13} \neq 0$, we get that (1) is isomorphic to

$$(d^5) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_3 + \alpha e_2, \\ [e_2, e_4, e_5, e_6] = e_3, \\ [e_1, e_4, e_5, e_6] = e_1, \end{cases} \qquad \alpha \in F, \alpha \neq 0.$$

by substituting $\frac{b^2_{12}}{b^2_{13}}e_3 + \frac{b^2_{12}}{b^3_{13}}e_2$, $\frac{1}{b^2_{13}}e_6$, $\frac{b^2_{12}}{b^3_{13}}e_2$ and $\frac{(b^2_{12})^2}{b^2_{13}b^3_{13}}e_1$ for e_3, e_6, e_2 and e_1 in $(1)'$.

In case $b_{12}^2 = b_{13}^3 = 0$, substituting $\frac{1}{b_{12}^3}e_6$ for e_6 in $(1)'$, we get

$$(d^6)\quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_3, \\ [e_2, e_4, e_5, e_6] = \gamma e_2, \\ [e_1, e_4, e_5, e_6] = (\gamma + 1)e_1, \end{cases} \qquad \gamma \in F, \quad \gamma \neq 0, \quad \gamma = -1.$$

Similarly, in case $b_{12}^2 = 0$ and $b_{13}^3 \neq 0$, $(1)'$ is isomorphic to case (d^4) or (d^5).

Second, we analyze table (2). If we impose the Jacobi identities on (2), and substitute $e_6 + b_{15}^1 e_1 - b_{25}^1 e_2 - b_{45}^1 e_4 + b_{35}^1 e_3$ for e_6, then table (2) is reduced to

$$(2)'\quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^1 e_1 + b_{12}^2 e_2 + b_{12}^3 e_3, \\ [e_2, e_4, e_5, e_6] = b_{13}^1 e_1 + b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_2, e_3, e_5, e_6] = b_{14}^1 e_1, \\ [e_1, e_4, e_5, e_6] = b_{23}^1 e_1, \\ [e_1, e_3, e_5, e_6] = b_{24}^1 e_1, \\ [e_1, e_2, e_5, e_6] = b_{34}^1 e_1, \end{cases}$$

where $b_{13}^2 + b_{12}^3 = 0$, $\Delta = \det\begin{pmatrix} b_{12}^2 & b_{12}^3 \\ b_{13}^2 & b_{13}^3 \end{pmatrix} \neq 0$. And

$$\begin{cases} e_2 = \dfrac{[b_{13}^3 e_3 - b_{12}^3 e_2, e_4, e_5, e_6] + (b_{12}^3 b_{13}^1 - b_{13}^3 b_{12}^1)e_1}{\Delta}, \\ e_3 = \dfrac{[b_{12}^2 e_2 - b_{13}^2 e_3, e_4, e_5, e_6] + (b_{12}^1 b_{13}^2 - b_{12}^2 b_{13}^1)e_1}{\Delta}, \end{cases}$$

$$[e_2, e_1, e_3, e_4] = \frac{\left[\left[b_{13}^3 e_3 - b_{12}^3 e_2, e_4, e_5, e_6\right] + \left(b_{12}^3 b_{13}^1 - b_{13}^3 b_{12}^1\right)e_1, e_1, e_3, e_4\right]}{\Delta}$$

$$= \frac{\left[b_{12}^3 e_1, e_4, e_5, e_6\right]}{\Delta} = \frac{b_{12}^3 b_{23}^1}{\Delta} e_1 = -e_1,$$

$$[e_3, e_1, e_2, e_4] = \frac{\left[\left[b_{12}^2 e_2 - b_{13}^2 e_3, e_4, e_5, e_6\right] + \left(b_{12}^1 b_{13}^2 - b_{12}^2 b_{13}^1\right)e_1, e_1, e_2, e_4\right]}{\Delta}$$

$$= \frac{\left[-b_{13}^2 e_1, e_4, e_5, e_6\right]}{\Delta} = \frac{-b_{13}^2 b_{23}^1}{\Delta} e_1 = e_1.$$

It follows that $\Delta = -b_{13}^2 b_{23}^1 = -b_{12}^3 b_{23}^1$. Thanks to $b_{13}^2 + b_{12}^3 = 0$, we get $b_{13}^2 = 0$ and $b_{23}^1 = 0$. This contradicts $\Delta \neq 0$. Therefore case (2) is not realized.

In similar discussions to case (2), we find that cases (3) and (4) are also not realized.

Now we study table (5). Imposing the Jacobi identities on (5) and substituting $e_6 + b_{12}^2 e_1 + b_{12}^1 e_2 - b_{13}^1 e_3 + b_{14}^1 e_4 - b_{15}^1 e_5$ for e_6, we get

$$(5)'\quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_2, e_3, e_5, e_6] = b_{14}^2 e_2, \\ [e_2, e_3, e_4, e_6] = b_{15}^2 e_2, \\ [e_1, e_3, e_5, e_6] = b_{14}^2 e_1, \\ [e_1, e_3, e_4, e_6] = b_{15}^2 e_1. \end{cases}$$

Again from

$$-b_{14}^2 e_2 = [e_3, e_2, e_5, e_6] = [[e_1, e_2, e_4, e_5], e_2, e_5, e_6] = 0,$$
$$-b_{15}^2 e_2 = [e_3, e_2, e_4, e_6] = [[e_1, e_2, e_4, e_5], e_2, e_4, e_6] = 0,$$

we have $b_{14}^2 = b_{15}^2 = 0$. Therefore (5) is isomorphic to

$$(d^7) \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3. \end{cases}$$

It is trivial that cases $(d^1), \ldots, (d^7)$ represent distinct classes.

(5) If $\dim A^1 = 4$, let $A^1 = Fe_1 + Fe_2 + Fe_3 + Fe_4$. Then the multiplication table of A in a basis $e_1, \ldots, e_6$ has only the following possibilities

$$(1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, e_2, e_3, e_5] = e_4, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

$$(2) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

$$(3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

$$(4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

$$(5) \quad \begin{cases} [e_1, e_2, e_3, e_4] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

$$(6) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^4 b_{ij}^k e_k, \end{cases}$$

where $b_{ij}^k \in F, 1 \leqslant i < j \leqslant 5$.

First, imposing the Jacobi identities on table (1) for $\{e_3, e_4, e_5, e_6\}$, $\{e_2, e_4, e_5, e_6\}$, $\{e_2, e_3, e_5, e_6\}$, $\{e_2, e_4, e_5, e_6\}$, $\{e_1, e_3, e_5, e_6\}$, $\{e_1, e_4, e_5, e_6\}$, $\{e_1, e_3, e_4, e_6\}$, $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_2, e_4, e_6\}$, $\{e_1, e_2, e_3, e_6\}$ and substituting $e_6 + b_{12}^2 e_1 + b_{12}^1 e_2 - b_{13}^1 e_3 + b_{14}^1 e_4 - b_{15}^1 e_5$ for e_6, we find that (1) is isomorphic to

$$(e^1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, e_2, e_3, e_5] = e_4. \end{cases}$$

If we impose Jacobi identities on tables (2)–(5) respectively, then we get $\dim A^1 < 4$. This contradicts $\dim A = 4$. Therefore cases (2)–(5) are not realized.

Finally, we study table (6). Imposing the Jacobi identities on table (6) for $\{e_1, e_4, e_5, e_6\}$, $\{e_1, e_3, e_5, e_6\}$, $\{e_1, e_3, e_4, e_6\}$, $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_2, e_4, e_6\}$, $\{e_1, e_2, e_3, e_6\}$ and substituting $e_6 + b_{12}^1 e_2 - b_{13}^1 e_3 + b_{14}^1 e_4 - b_{15}^1 e_5$ for e_6, we get

$$(6)' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^2 e_2 + b_{12}^3 e_3 + b_{12}^4 e_4, \\ [e_2, e_4, e_5, e_6] = b_{13}^2 e_2 + b_{13}^3 e_3 + b_{13}^4 e_4, \\ [e_2, e_3, e_5, e_6] = b_{14}^2 e_2 + b_{14}^3 e_3 + b_{14}^4 e_4, \\ [e_2, e_3, e_4, e_6] = b_{15}^2 e_2 + b_{15}^3 e_3 + b_{15}^4 e_4, \\ [e_1, e_4, e_5, e_6] = \left(b_{13}^2 + b_{12}^3\right) e_1, \\ [e_1, e_3, e_5, e_6] = \left(b_{14}^2 - b_{12}^4\right) e_1, \\ [e_1, e_3, e_4, e_6] = b_{15}^2 e_1, \\ [e_1, e_2, e_5, e_6] = \left(-b_{14}^3 - b_{13}^4\right) e_1, \\ [e_1, e_2, e_4, e_6] = -b_{15}^3 e_1, \\ [e_1, e_2, e_3, e_6] = b_{15}^4 e_1, \end{cases}$$

and $b_{12}^2 b_{15}^3 = b_{12}^3 b_{15}^2, b_{13}^2 b_{15}^3 = b_{13}^3 b_{15}^2, b_{14}^2 b_{15}^4 = b_{14}^4 b_{15}^2$, $b_{12}^2 b_{15}^4 = b_{12}^4 b_{15}^2, b_{13}^3 b_{15}^4 = b_{15}^3 b_{13}^4$, $b_{14}^3 b_{15}^4 = b_{15}^3 b_{14}^4$. Since

$$\dim A^1 = 4, \qquad \det \begin{pmatrix} b_{12}^2 & b_{12}^3 & b_{12}^4 \\ b_{13}^2 & b_{13}^3 & b_{13}^4 \\ b_{14}^2 & b_{14}^3 & b_{14}^4 \end{pmatrix} \neq 0$$

and $b_{15}^2 = b_{15}^3 = b_{15}^4 = 0$. Therefore $(6)'$ is reduced to

$$(6)'' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^2 e_2 + b_{12}^3 e_3 + b_{12}^4 e_4, \\ [e_2, e_4, e_5, e_6] = b_{13}^2 e_2 + b_{13}^3 e_3 + b_{13}^4 e_4, \\ [e_2, e_3, e_5, e_6] = b_{14}^2 e_2 + b_{14}^3 e_3 + b_{14}^4 e_4, \\ [e_1, e_4, e_5, e_6] = \left(b_{13}^2 + b_{12}^3\right) e_1, \\ [e_1, e_3, e_5, e_6] = \left(b_{14}^2 - b_{12}^4\right) e_1, \\ [e_1, e_2, e_5, e_6] = \left(-b_{14}^3 - b_{13}^4\right) e_1. \end{cases}$$

Fix e_5 and e_6 in the operation of A and endow with a Lie operation

$$[x, y]_0 = [x, y, e_5, e_6], x, y \in A$$

on the vector space $A_0 = A$. Then A_0 is a Lie algebra. And the multiplication table of A_0 in the basis $e_1, \dots, e_6$ is as follows:

$$\begin{cases} [e_3, e_4]_0 = b_{12}^2 e_2 + b_{12}^3 e_3 + b_{12}^4 e_4, \\ [e_2, e_4]_0 = b_{13}^2 e_2 + b_{13}^3 e_3 + b_{13}^4 e_4, \\ [e_2, e_3]_0 = b_{14}^2 e_2 + b_{14}^3 e_3 + b_{14}^4 e_4, \\ [e_1, e_4]_0 = \left(b_{13}^2 + b_{12}^3\right) e_1, \\ [e_1, e_3]_0 = \left(b_{14}^2 - b_{12}^4\right) e_1, \\ [e_1, e_2]_0 = \left(-b_{14}^3 - b_{13}^4\right) e_1. \end{cases}$$

Therefore, $C = Fe_2 + Fe_3 + Fe_4$ is a simple subalgebra of A_0 and Fe_1 is a one-dimensional ideal of A_0. It follows that $[e_1, e_4]_0 = [e_1, e_3]_0 = [e_1, e_2]_0 = 0$. Therefore, $(6)''$ has a reduced form

$$(6)''' \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = b_{12}^2 e_2 + b_{12}^3 e_3 + b_{12}^4 e_4, \\ [e_2, e_4, e_5, e_6] = b_{13}^2 e_2 + b_{13}^3 e_3 + b_{13}^4 e_4, \\ [e_2, e_3, e_5, e_6] = b_{14}^2, e_2 + b_{14}^3 e_3 + b_{14}^4 e_4, \end{cases} \quad \text{and} \quad \det \begin{pmatrix} b_{12}^2 & b_{12}^3 & b_{12}^4 \\ b_{13}^2 & b_{13}^3 & b_{13}^4 \\ b_{14}^2 & b_{14}^3 & b_{14}^4 \end{pmatrix} \neq 0.$$

Now fixing e_6 in the operation of A, we get a 3-Lie algebra A_1 ($A_1 = A$ as vector spaces) with the 3-ary operation

$$[x, y, z]_1 = [x, y, z, e_6], x, y, z \in A_1.$$

From table $(6)'''$, $Fe_1 + Fe_6$ is the center of A_1, and $D = Fe_2 + Fe_3 + Fe_4 + Fe_5$ is a four-dimensional subalgebra of A_1 with $\dim D^1 = 3$. By theorem 3.1, (6) is isomorphic to

$$(e^2) \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_2, \\ [e_2, e_4, e_5, e_6] = e_3, \\ [e_2, e_3, e_5, e_6] = e_4. \end{cases}$$

It is trivial that (e^1) is not isomorphic to (e^2).

(6) If $\dim A^1 = 5$, let $A^1 = Fe_1 + Fe_2 + Fe_3 + Fe_4 + Fe_5$. Then the multiplication table of A in the basis $e_1, \ldots, e_6$ has only the following possibilities:

$$(1) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, e_2, e_3, e_5] = e_4, \\ [e_1, e_2, e_3, e_4] = e_5, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^5 b_{ij}^k e_k, \end{cases}$$

$$(2) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, e_2, e_3, e_5] = e_4, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^5 b_{ij}^k e_k, \end{cases}$$

$$(3) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, e_2, e_4, e_5] = e_3, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^5 b_{ij}^k e_k, \end{cases}$$

$$(4) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^5 b_{ij}^k e_k, \end{cases}$$

$$(5) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1 + \beta e_2, \\ [e_1, e_3, e_4, e_5] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^5 b_{ij}^k e_k, \end{cases}$$

$$(6)\quad \begin{cases}[e_1, e_2, e_3, e_4] = e_1,\\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{5} b_{ij}^k e_k,\end{cases}$$

$$(7)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_6] = \sum_{k=1}^{5} b_{ij}^k e_k,\end{cases}$$

where $b_{ij}^k \in F$, $1 \leqslant i < j \leqslant 5$.

By a similar discussion to that of case $\dim A^1 = 4$, we find that table (1) is isomorphic to

$$(f^1)\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_1, e_3, e_4, e_5] = e_2,\\ [e_1, e_2, e_4, e_5] = e_3,\\ [e_1, e_2, e_3, e_5] = e_4,\\ [e_1, e_2, e_3, e_4] = e_5.\end{cases}$$

And cases (2)–(6) are not realized.

Now we analyze table (7). If we impose the Jacobi identities on (7) for $\{e_1, e_4, e_5, e_6\}$, $\{e_1, e_3, e_5, e_6\}$, $\{e_1, e_3, e_4, e_6\}$, $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_2, e_4, e_6\}$, $\{e_1, e_2, e_3, e_6\}$ and substitute $e_6 + b_{12}^1 e_2 - b_{13}^1 e_3 + b_{14}^1 e_4 - b_{15}^1 e_5$ for e_6, then (7) is reduced to

$$(7)'\quad \begin{cases}[e_2, e_3, e_4, e_5] = e_1,\\ [e_3, e_4, e_5, e_6] = b_{12}^2 e_2 + b_{12}^3 e_3 + b_{12}^4 e_4 + b_{12}^5 e_5,\\ [e_2, e_4, e_5, e_6] = b_{13}^2 e_2 + b_{13}^3 e_3 + b_{13}^4 e_4 + b_{13}^5 e_5,\\ [e_2, e_3, e_5, e_6] = b_{14}^2 e_2 + b_{14}^3 e_3 + b_{14}^4 e_4 + b_{14}^5 e_5,\\ [e_2, e_3, e_4, e_6] = b_{15}^2 e_2 + b_{15}^3 e_3 + b_{15}^4 e_4 + b_{15}^5 e_5,\\ [e_1, e_4, e_5, e_6] = \left(b_{12}^3 + b_{13}^2\right) e_1,\\ [e_1, e_3, e_5, e_6] = \left(b_{14}^2 - b_{12}^4\right) e_1,\\ [e_1, e_3, e_4, e_6] = \left(b_{12}^5 + b_{15}^2\right) e_1,\\ [e_1, e_2, e_5, e_6] = \left(-b_{14}^3 - b_{13}^4\right) e_1,\\ [e_1, e_2, e_4, e_6] = \left(b_{13}^5 - b_{15}^3\right) e_1,\\ [e_1, e_2, e_3, e_6] = \left(b_{14}^5 + b_{15}^4\right) e_1.\end{cases}$$

Since

$$\dim A^1 = 5, \qquad \Delta = \det \begin{pmatrix} b_{12}^2 & b_{12}^3 & b_{12}^4 & b_{12}^5 \\ b_{13}^2 & b_{13}^3 & b_{13}^4 & b_{13}^5 \\ b_{14}^2 & b_{14}^3 & b_{14}^4 & b_{14}^5 \\ b_{15}^2 & b_{15}^3 & b_{15}^4 & b_{15}^5 \end{pmatrix} \neq 0.$$

Fixing e_6 in the operation of A, we get a six-dimensional 3-Lie algebra A_1 ($A_1 = A$ as vector spaces) with a 3-ary operation as follows:

$$[x, y, z]_1 = [x, y, z, e_6], x, y, z \in A_1.$$

From $(7)'$ and $\Delta \neq 0$, we have $A_1 = E \oplus Z(A_1)$, where $E = Fe_2 + Fe_3 + Fe_4 + Fe_5$ is a four-dimensional subalgebra of A_1, $\dim E^1 = 4$ and $Z(A_1) = Fe_1 + Fe_6$. Then we can choose a suitable basis of E, which is still denoted by e_2, e_3, e_4, e_5, and a suitable vector e_1 such that (7) can be reduced to the form

$$(f^2) \quad \begin{cases} [e_2, e_3, e_4, e_5] = e_1, \\ [e_3, e_4, e_5, e_6] = e_2, \\ [e_2, e_4, e_5, e_6] = e_3, \\ [e_2, e_3, e_5, e_6] = e_4, \\ [e_2, e_3, e_4, e_6] = e_5. \end{cases}$$

It is trivial that (f^1) is not isomorphic to (f^2). □

Acknowledgment

R Bai would like to thank Dr Y Zhang and Dr S Zhao (University of Queensland, Australia) for their useful suggestions. R Bai would like to thank the Nankai Institute of Mathematics too. Part of this work was done here when she visited Nankan University in 2007.

References

[1] Nambu Y 1973 Generalized hamiltonian dynamics *Phys. Rev.* D **7** 2405–12
[2] Takhtajan L 1994 On foundation of the generalized Nambu mechanics *Commun. Math. Phys.* **160** 295–315
[3] Filippov V T 1985 *n*-Lie algebras *Sib. Mat. Zh.* **26** 126–40
[4] Bagger J and Lambert N 2008 Gauge symmetry and supersymmetry of multiple $M2$-branes *Phys. Rev.* D **77** 065008 (arXiv:0711.0955 [hep-th])
[5] Ho P, Hou R and Matsuo Y 2008 Lie 3-algebra and multiple M_2-branes (arXiv:0804.2110v2 [hep-th])
[6] Ho P, Chebotar M and Ke W 2003 On skew-symmetric maps on Lie algebras *Proc. R. Soc. Edinb.* **133A** 1273–81
[7] Papadopoulos G 2008 $M2$-branes, 3-Lie algebras and Plucker relations arXiv:0804.2662 [hep-th]
[8] Alekseevsky D and Guha P 1996 On decomposability of Nambu–Poisson tensor *Acta Math. Univ. Comen.* **65** 1–9
[9] Gautheron P 1998 Simple facts concerning Nambu algebras *Commun. Math. Phys.* **195** 417–34
[10] Michor P W and Vinogradov A M 1996 *n*-ary and associative algebras *Rend. Sem. Mat. Univ. Pol. Torino* **53** 373–92
[11] Marmo G, Vilasi G and Vinogradov A M 1998 The local structure of *n*-Poisson and *n*-Jacobi manifolds *J. Geom. Phys.* **25** 141–82
[12] Nakanishi N 1998 On Nambu–Poisson manifolds *Rev. Math. Phys.* **10** 499–510
[13] Vinogradov A and Vinogradov M 1998 On multiple generalizations of lie algebras and poisson manifolds *Am. Math. Soc., Contemp. Math.* **219** 273–87
[14] Ling W 1993 On the structure of *n*-Lie algebras, Dissertation *University-GHS-Siegen, Siegn*
[15] Pozhidaev A P 1998 Simple quotient algebras and subalgebras of Jacobian algebras *Sib. Math. J.* **39** 512–7
[16] Pozhidaev A P 1999 Two classes of central simple *n*-Lie algebras *Sib. Math. J.* **40** 1112–8
[17] Bai R, Wang X, Xiao W and An H 2008 The structure of low dimensional *n*-Lie algebras over a field of characteristic 2 *Linear Algebr. Appl.* **428** 1912–20
[18] Bai R and Meng D 2003 The strong semi-simple *n*-Lie algebras *Commun. Algebr.* **31** 5331–41
[19] Barnes D W 2007 On $(n+2)$-dimensional *n*-Lie algebras arXiv:0704.1892
[20] Bai R and Meng D 2006 Representations of strong semisimple *n*-Lie algebras *Adv. Math. (China)* **35** 739–46
[21] Figueroa-Ofarrill J 2008 Lorenzian Lie *n*-algebras arXiv:0805.4760v3 [math.Rt])